Congruences involving Franel and Catalan-Larcombe-French numbers

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Abstract

Let $\{f_n\}$ be the Franel numbers given by $f_n=\sum_{k=0}^n\binom nk^3$, and let $p>5$ be a prime. In this paper we mainly determine $\sum_{k=0}^{p-1}\binom{2k}k\frac{f_k}{m^k}\pmod p$ for $m=5,-16,16,32,-49,50,96$. Let $S_n=\sum_{k=0}^n\binom nk\binom{2k}k\binom{2n-2k}{n-k}$. We also determine $\sum_{k=0}^{p-1}\binom{2k}k\frac{S_k}{m^k}\pmod p$ for $m=7,16,25,32,64,160,800,1600,156832$.

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1. Introduction

Let $[x]$ be the greatest integer not exceeding x, and let $(\frac ap)$ be the Legendre symbol. For a prime p let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. For positive integers a,b and n, if $n=ax^2+by^2$ for some integers x and y, we briefly write that $n=ax^2+by^2$.

In 1894 J. Franel [F] introduced the following Franel numbers $\{f_n\}$:

$$f_n=\sum_{k=0}^n\binom nk^3\quad(n=0,1,2,\ldots).$$

The first few Franel numbers are as below:

$$f_0=1,\ f_1=2,\ f_3=10,\ f_4=56,\ f_5=346,\ f_6=2252,\ f_7=15184.$$

It is known that

$$(n+1)^2f_{n+1}=(7n^2+7n+2)f_n+8n^2f_{n-1}\ (n\ge1).$$

Let p be an odd prime and $m\in\mathbb{Z}$ with $m\not\equiv0\pmod p$. In [S6], the author made many conjectures on $\sum_{k=0}^{p-1}\binom{2k}k\frac{f_k}{m^k}\pmod{p^2}$. For example, for any odd prime p,

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$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-16)^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\equiv 1 \pmod{12} \text{ with } 6\mid y,\\ 2p-4x^2 \pmod{p^2} & \text{if } p=x^2+y^2\equiv 1 \pmod{12} \text{ with } 6\mid x-3,\\ 4(\frac{xy}{3})xy \pmod{p^2} & \text{if } p=x^2+y^2\equiv 5 \pmod{12},\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

In [Gu], J.W. Guo proved that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-16)^k}\equiv 0 \pmod p \quad \text{for} \quad p\equiv 3 \pmod 4$$

and

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-16)^k}\binom{2k}{k}f_k\equiv(-1)^{\frac{p-1}{2}}p \pmod{p^3},$$

where the second congruence modulo p^2 was conjectured by the author in [S6]. We note that $p\mid\binom{2k}{k}$ for $k=\frac{p+1}{2},\dots,p-1$. In [Su4, Su5], the author's brother Z.W. Sun investigated congruences for Franel numbers. In particular, he showed that for any odd prime p,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-4)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k} \pmod{p^2}.$$

By [S3, Theorems 3.3 and 3.4],

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and so } p=x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Thus,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-4)^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and so } p=x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

For any nonnegative integer n let

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2,\quad D_n=\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2,$$

$$(1.1)\qquad a_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k},\quad b_n=\sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}\binom{n+k}{k}(-3)^{n-3k},$$

$$S_n=\frac{P_n}{2^n}=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}=\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}^2 4^{n-2k}.$$

Here $\{A_n\}$ is called Apéry numbers since Apéry [Ap] used it to prove $\zeta(3)$ is irrational in 1979, $\{D_n\}$ is called Domb numbers, $\{b_n\}$ is called Almkvist-Zudilin numbers, and $\{P_n\}$

is called Catalan-Larcombe-French numbers. See [CCL], [CV], [Co], [D], [JV], [Su6] and [Z]. Such sequences appear as coefficients in various series for $1/\pi$, for example,

$$\sum_{k=0}^{\infty}\frac{9k+2}{50^k}\binom{2k}{k}f_k=\frac{25}{2\pi},\quad \sum_{k=0}^{\infty}\frac{5k+1}{64^k}D_k=\frac{8}{\sqrt{3}\pi},\quad \sum_{n=0}^{\infty}\frac{4k+1}{81^k}b_k=\frac{3\sqrt{3}}{2\pi}.$$

Let $p>3$ be a prime, $m\in\mathbb{Z}_p$ and $m\not\equiv 0,-4,-8 \pmod p$. In this paper, we show that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{m}{(m+8)^2}\Big)^k f_k\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{m}{(m+4)^3}\Big)^k \pmod p.$$

Let $x\in\mathbb{Z}_p$, $x\not\equiv 0,-1,-\frac13 \pmod p$ and $\big(\frac{9x^2+14x+9}{p}\big)=1$. We also show that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9x^2+14x+9}\Big)^k f_k\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{x}{9(1+3x)^4}\Big)^k \pmod p.$$

As consequences we determine $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{n^k} \pmod p$ for $n=5,-16,16,32,-49,50,96$ and $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{4^k} \pmod p$. As examples, for any prime $p>5$ we have

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-16)^k}\equiv 4x^2 \pmod p\quad\text{for}\quad p=x^2+9y^2,$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{16^k}\equiv 4x^2 \pmod p\quad\text{for}\quad p=x^2+5y^2,$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{5^k}\equiv 4x^2 \pmod p\quad\text{for}\quad p=x^2+15y^2,$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{32^k}\equiv 4x^2 \pmod p\quad\text{for}\quad p=x^2+6y^2.$$

Thus we partially solve some conjectures in [S6].

In [Su6] Z.W. Sun introduced

$$S_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}x^k\quad(n=0,1,2,\ldots)$$

and used it to establish new series for $1/\pi$. Note that $S_n(1)=S_n$ is essentially the Catalan-Larcombe-French number. In [JV], Jarvis and Verrill gave some congruences for $P_n=2^nS_n$. In Section 3 we establish some new identities involving S_n. For example,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{S_k}{8^k}=\frac{S_n}{8^n}\quad\text{and}\quad\sum_{k=0}^{2n}\binom{2n}{k}\binom{2n+k}{k}(-8)^{2n-k}S_k=(-1)^n\binom{2n}{n}^3.$$

Let p be an odd prime, $n\in\mathbb{Z}_p$ and $n\not\equiv 0,-16 \pmod p$. In Section 3 we also prove that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(n+16)^k}\equiv\Big(\frac{n(n+16)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{n^{2k}} \pmod p.$$

As consequences we determine $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{m^k} \pmod p$ for $m=7,16,25,32,64,160,800,1600,156832$. For example, for any prime $p>7$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{7^k}\equiv\begin{cases}4x^2 \pmod p & \text{if } p\equiv 1,2,4 \pmod 7 \text{ and so } p=x^2+7y^2,\\ 0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7).\end{cases}$$

Let p be an odd prime, $n,x\in\mathbb{Z}_p$ and $n(n+4x)\not\equiv 0 \pmod p$. In Section 4 we show that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(x)}{(n+4x)^k}\equiv\Big(\frac{n(n+4x)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{n^{3k}} \pmod p,$$

where

$$C_n(x)=\sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}x^{n-3k}.$$

In this paper we also pose some conjectures for congruences involving f_n or S_n. See Conjectures 2.1-2.2 and Conjectures 3.1-3.4.

2. Congruences involving $\{f_n\}$

Lemma 2.1. *Let p be an odd prime, $u\in\mathbb{Z}_p$ and $u\not\equiv 1 \pmod p$. For any p-adic sequences $\{c_k\}$ we have*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k c_k\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}c_k \pmod p.$$

Proof. Note that $\binom{-x}{k}=(-1)^k\binom{x+k-1}{k}$ and $\binom{n}{k}\binom{n+k}{k}=\binom{2k}{k}\binom{n+k}{2k}$. Using Fermat's little theorem we deduce that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k c_k\\ &\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}c_ku^k(1-u)^{p-1-2k}=\sum_{k=0}^{(p-1)/2}\binom{2k}{k}c_ku^k\sum_{r=0}^{p-1-2k}\binom{p-1-2k}{r}(-u)^r\\ &=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}c_k(-1)^{n-k}\binom{p-1-2k}{n-k}=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}c_k\binom{n+k-p}{n-k}\\ &\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{2k}{k}c_k\binom{n+k}{n-k}=\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}c_k \pmod p.\end{aligned}$$

Thus the lemma is proved.

Lemma 2.2. *Let $p>3$ be a prime and $c_0,c_1,\ldots,c_{p-1}\in\mathbb{Z}_p$. Then*

$$\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}c_k\equiv\sum_{k=0}^{p-1}\frac{p}{2k+1}(-1)^kc_k \pmod{p^3}.$$

Proof. Since

$$\begin{aligned}\sum_{k=0}^{m}\binom{x}{k}(-1)^k&=\sum_{k=0}^{m}\binom{x-1}{k}(-1)^k+\sum_{k=1}^{m}\binom{x-1}{k-1}(-1)^k\\&=\sum_{r=0}^{m}\binom{x-1}{r}(-1)^r-\sum_{r=0}^{m-1}\binom{x-1}{r}(-1)^r\\&=\binom{x-1}{m}(-1)^m=\binom{m-x}{m},\end{aligned}$$

and $\binom{n}{k}\binom{n+k}{k}=\binom{2k}{k}\binom{n+k}{2k}$ we see that

$$\begin{aligned}&\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}c_k\\&=\sum_{k=0}^{p-1}\sum_{n=k}^{p-1}\binom{2k}{k}\binom{n+k}{2k}c_k=\sum_{k=0}^{p-1}\binom{2k}{k}c_k\sum_{r=0}^{p-1-k}\binom{2k+r}{2k}\\&=\sum_{k=0}^{p-1}\binom{2k}{k}c_k\sum_{r=0}^{p-1-k}\binom{-2k-1}{r}(-1)^r=\sum_{k=0}^{p-1}\binom{2k}{k}c_k\binom{p+k}{p-1-k}\\&=\sum_{k=0}^{p-1}\binom{2k}{k}c_k\binom{p+k}{2k+1}=\sum_{k=0}^{p-1}\frac{p}{2k+1}c_k\frac{(p^2-1^2)\cdots(p^2-k^2)}{k!^2}.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}c_k\\&=\frac{(p^2-1^2)\cdots(p^2-(\frac{p-1}{2})^2)}{(\frac{p-1}{2})!^2}c_{\frac{p-1}{2}}+p\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\frac{c_k}{2k+1}\cdot\frac{(p^2-1^2)\cdots(p^2-k^2)}{k!^2}\\&\equiv(-1)^{\frac{p-1}{2}}\Big(1-p^2\sum_{r=1}^{(p-1)/2}\frac{1}{r^2}\Big)c_{\frac{p-1}{2}}+p\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\frac{(-1)^kc_k}{2k+1}\pmod{p^3}.\end{aligned}$$

It is known that $\sum_{r=1}^{(p-1)/2}\frac{1}{r^2}\equiv 0\pmod p$. Thus the result follows.

Example 2.1. Let $\{P_n(x)\}$ be the famous Legendre polynomials. Then Murphy proved that

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\Big(\frac{x-1}{2}\Big)^k.$$

Thus, applying Lemma 2.2 we see that for any prime $p>3$ and $x\in\mathbb{Z}_p$,

$$\sum_{n=0}^{p-1}P_n(x)\equiv\sum_{k=0}^{p-1}\frac{p}{2k+1}\Big(\frac{1-x}{2}\Big)^k\pmod{p^3}.\tag{2.1}$$

Lemma 2.3 ([CTYZ, (2.19), p.1305 and (2.27)]. *Let n be a nonnegative integer. Then*

$$A_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} f_k = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n-k} a_k$$

and

$$\frac{D_n}{8^n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{f_k}{(-8)^k}.$$

Lemma 2.3 can be verified straightforward by using Maple and the method in [CHM] to compare the recurrence relations for both sides.

Theorem 2.1. *Let p be an odd prime, $m \in \mathbb{Z}_p$ and $m \not\equiv 0, -4, -8 \pmod p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}\Big(\frac{m}{(m+8)^2}\Big)^k f_k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2\binom{3k}{k}\Big(\frac{m}{(m+4)^3}\Big)^k \pmod p.$$

Proof. Taking $c_k = \frac{f_k}{(-8)^k}$ in Lemma 2.1 and then applying Lemma 2.3 we see that for $u \in \mathbb{Z}_p$ with $u \not\equiv 1 \pmod p$,

$$\sum_{k=0}^{p-1} \binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k \frac{f_k}{(-8)^k} \equiv \sum_{n=0}^{p-1} u^n \frac{D_n}{8^n} \pmod p.$$

Now substituting u with $-\frac{8}{m}$ in the above formula we deduce that

$$\sum_{k=0}^{p-1} \binom{2k}{k}\Big(\frac{m}{(m+8)^2}\Big)^k f_k \equiv \sum_{n=0}^{p-1} \frac{D_n}{(-m)^n} \pmod p. \tag{2.2}$$

By [S8, Theorem 3.1],

$$\sum_{n=0}^{p-1} \frac{D_n}{(-m)^n} \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2\binom{3k}{k}\Big(\frac{m}{(m+4)^3}\Big)^k \pmod p.$$

Thus the theorem is proved.

Theorem 2.2. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}\frac{f_k}{50^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2+3y^2, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Proof. Taking $m = 2$ in Theorem 2.1 we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k}\frac{f_k}{50^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \pmod p.$$

From [M] and [Su2] we know that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2+3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Thus the result follows.

Theorem 2.3. *Let p be a prime with $p\equiv\pm1\pmod 8$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{32^k}\equiv\begin{cases}4x^2\pmod p & \text{if } p\equiv 1,7\pmod{24} \text{ and so } p=x^2+6y^2,\\ 0\pmod p & \text{if } p\equiv 17,23\pmod{24}.\end{cases}$$

Proof. Taking $m=8$ in Theorem 2.1 we see that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{32^k}\equiv\sum_{n=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}\pmod p.$$

Now applying [S2, Theorem 4.5] we deduce the result.

Theorem 2.4. *Let p be a prime with $p\equiv\pm1\pmod 5$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-49)^k}\equiv\begin{cases}4x^2\pmod p & \text{if } p\equiv 1,19\pmod{30} \text{ and so } p=x^2+15y^2,\\ 0\pmod p & \text{if } p\equiv 11,29\pmod{30}.\end{cases}$$

Proof. Taking $m=-1$ in Theorem 2.1 we see that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-49)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\pmod p.$$

Now applying [S2, Theorem 4.6] we deduce the result.

Theorem 2.5. *Let p be a prime such that $p\equiv 1,19\pmod{30}$ and so $p=x^2+15y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{5^k}\equiv 4x^2\pmod p.$$

Proof. Let $t\in\{1,2,\ldots,\frac{p-1}{2}\}$ be given by $t^2\equiv-15\pmod p$ and $m=(-11+3t)/2$. Then $\frac{64}{m}\equiv\frac{-11-3t}{2}\pmod p$ and so

$$\frac{(m+8)^2}{m}=16+m+\frac{64}{m}\equiv 16+\frac{-11+3t}{2}+\frac{-11-3t}{2}=5\pmod p.$$

We also have

$$\frac{(m+4)^3}{m}\equiv\frac{(\frac{-3+3t}{2})^3}{\frac{-11+3t}{2}}\equiv-27\pmod p.$$

Now applying Theorem 2.1 and [S2, Theorem 4.6] we deduce that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{5^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\equiv 4x^2\pmod p.$$

This proves the theorem.

Remark 2.1 Let p be an odd prime. Taking $m=-16$ in Theorem 2.1 we deduce the congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-4)^k}\pmod p$.

Theorem 2.6. *Let p be an odd prime and $u\in\mathbb{Z}_p$.*

(i) *If* $u \not\equiv 1 \pmod p$, *then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k f_k \equiv \sum_{n=0}^{p-1} A_n u^n \pmod p.$$

(ii) *If* $u \not\equiv -1 \pmod p$, *then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1+u)^2}\Big)^k a_k \equiv \sum_{n=0}^{p-1} A_n u^n \pmod p.$$

Proof. Taking $c_k = f_k$ in Lemma 2.1 and then applying Lemma 2.3 we obtain (i). Taking $c_k = (-1)^k a_k$ in Lemma 2.1 and then applying Lemma 2.3 we see that for $u \not\equiv 1 \pmod p$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k (-1)^k a_k \equiv \sum_{n=0}^{p-1} u^n \cdot (-1)^n A_n \pmod p.$$

Now substituting u with $-u$ in the above we deduce (ii), which completes the proof.

Theorem 2.7. *Let* $p > 3$ *be a prime,* $x \in \mathbb{Z}_p$, $x \not\equiv 0, -1, -\frac13 \pmod p$ *and* $\big(\frac{9x^2+14x+9}{p}\big) = 1$. *Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9x^2+14x+9}\Big)^k f_k \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{x}{9(1+3x)^4}\Big)^k \pmod p.$$

Proof. Let $v \in \{1, 2, \ldots, \frac{p-1}{2}\}$ be given by $v^2 \equiv 9x^2 + 14x + 9 \pmod p$, and let

$$u = \frac{2x + v^2 + 3(x+1)v}{2x}.$$

Then $u \in \mathbb{Z}_p$. Since $v^2 \equiv 9x^2+14x+9 \not\equiv 9(x+1)^2 \pmod p$ we have $v \not\equiv \pm 3(x+1) \pmod p$. Thus $u \not\equiv 1 \pmod p$. If $u \equiv -1 \pmod p$, then $v^2 + 3(x+1)v \equiv -4x \pmod p$ and so $9(x+1)^2 \equiv v^2 + 4x \equiv -3(x+1)v \pmod p$. As $x + 1 \not\equiv 0 \pmod p$ we have $v \equiv -3(x+1) \pmod p$. We get a contradiction. Thus $u \not\equiv -1 \pmod p$. Note that

$$\begin{aligned}
&\frac{2x + v^2 + 3(x+1)v}{2x} \cdot \frac{2x + v^2 - 3(x+1)v}{2x} \\
&= \frac{(2x+v^2)^2 - 9(x+1)^2v^2}{4x^2} \equiv \frac{(9x^2+16x+9)^2 - 9(x+1)^2(9x^2+14x+9)}{4x^2} \\
&= \frac{(9x^2+16x+9)^2 - (9x^2+16x+9+2x)(9x^2+16x+9-2x)}{4x^2} = 1 \pmod p.
\end{aligned}$$

We see that $u \not\equiv 0 \pmod p$ and

$$u + \frac1u \equiv \frac{2x+v^2+3(x+1)v}{2x} + \frac{2x+v^2-3(x+1)v}{2x} = \frac{2x+v^2}{x} \equiv \frac{9x^2+16x+9}{x} \pmod p.$$

Now, from the above and Theorem 2.6 we deduce that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9x^2+14x+9}\Big)^k f_k$$

$$\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(u+\frac{1}{u}-2)^k}=\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k f_k$$
$$\equiv \sum_{n=0}^{p-1}A_nu^n\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1+u)^2}\Big)^k a_k$$
$$\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{(u+\frac{1}{u}+2)^k}=\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9(x+1)^2}\Big)^k a_k \pmod p.$$

Taking $u=\frac{x}{9}$ in [S8, Theorem 4.1] we see that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9(x+1)^2}\Big)^k a_k\equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{x}{9(1+3x)^4}\Big)^k \pmod p.$$

Thus the result follows.

Theorem 2.8. *Let p be a prime of the form $12k+1$ and so $p=x^2+9y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-16)^k}\equiv 4x^2 \pmod p.$$

Proof. Taking $x=-3$ in Theorem 2.7 we see that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-16)^k}\equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12288)^k} \pmod p.$$

Now applying [S3, Theorem 5.3] we deduce the result.

Theorem 2.9. *Let $p>5$ be a prime such that $p\equiv 1,5,19,23 \pmod{24}$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{96^k}\equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,19 \pmod{24} \text{ and so } p=x^2+2y^2,\\ 0 \pmod p & \text{if } p\equiv 5,23 \pmod{24}.\end{cases}$$

Proof. Since $(\frac{6}{p})=1$, taking $x=9$ in Theorem 2.7 we see that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{96^k}\equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \pmod p.$$

Now applying [S8, Theorem 5.6] we deduce the result.

Theorem 2.10. *Let p be a prime such that $p\equiv 1,9 \pmod{20}$ and so $p=x^2+5y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{16^k}\equiv 4x^2 \pmod p.$$

Proof. Let $t\in\{1,2,\ldots,\frac{p-1}{2}\}$ be given by $t^2\equiv -5 \pmod p$, and $x=\frac{1+4t}{9}$. Then $x\not\equiv 0,-1,-\frac13 \pmod p$, $\frac1x\equiv\frac{1-4t}{9} \pmod p$ and so $\frac{9x^2+14x+9}{x}=14+9(x+\frac1x)=16$. Thus, $(\frac{9x^2+14x+9}{p})=(\frac{16x}{p})=(\frac{1+4t}{p})=(\frac{-1-4t}{p})=(\frac{(2-t)^2}{p})=1$. We also have $\frac{9(1+3x)^4}{x}\equiv -1024 \pmod p$. Thus applying Theorem 2.7 and [S3, Theorem 5.5] we deduce that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{16^k}\equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k}\equiv 4x^2 \pmod p.$$

This proves the theorem.

Theorem 2.11. *Let $p>3$ be a prime and $z\in\mathbb{Z}_p$ with $z\not\equiv\frac14 \pmod p$. Then*

$$\sum_{n=0}^{p-1} f_n z^n \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\Big(\frac{z}{(1-4z)^3}\Big)^k \pmod p.$$

Proof. From [Su4, (2.3)] we know that

$$f_n=\sum_{k=0}^{n}\binom{2k}{k}\binom{3k}{k}\binom{n+2k}{3k}(-4)^{n-k}.$$

Thus,

$$\begin{aligned}\sum_{n=0}^{p-1} f_n z^n &= \sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{2k}{k}\binom{3k}{k}\binom{n+2k}{3k}(-4)^{n-k}z^n\\ &=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k\sum_{n=k}^{p-1}\binom{n+2k}{3k}(-4z)^{n-k}\\ &=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k\sum_{r=0}^{p-1-k}\binom{3k+r}{3k}(-4z)^{r}\\ &=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k\sum_{r=0}^{p-1-k}\binom{-3k-1}{r}(4z)^{r}\\ &\equiv\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k\sum_{r=0}^{p-1-k}\binom{p-1-3k}{r}(4z)^{r}\\ &=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k(1-4z)^{p-1-3k}\\ &\equiv\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\Big(\frac{z}{(1-4z)^3}\Big)^k \pmod p.\end{aligned}$$

This proves the theorem.

Similarly, from the formula (see [Su4, (2.2)])

$$f_n=\sum_{k=0}^{[n/2]}\binom{n+k}{3k}\binom{2k}{k}\binom{3k}{k}2^{n-2k}$$

we deduce the following result.

Theorem 2.12. *Let $p>3$ be a prime and $z\in\mathbb{Z}_p$ with $z\not\equiv\frac12 \pmod p$. Then*

$$\sum_{n=0}^{p-1} f_n z^n \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\Big(\frac{z^2}{(1-2z)^3}\Big)^k \pmod p.$$

Taking $c_k=f_k,(-1)^k a_k,\frac{f_k}{(-8)^k}$ in Lemma 2.2 and then applying Lemma 2.3 we see that for any prime $p>3$,

$$\sum_{n=0}^{p-1} A_n \equiv \sum_{k=0}^{p-1}\frac{p}{2k+1}(-1)^k f_k \pmod{p^3}, \tag{2.3}$$

$$\sum_{n=0}^{p-1}(-1)^n A_n \equiv \sum_{k=0}^{p-1}\frac{p}{2k+1}a_k \pmod{p^3}, \tag{2.4}$$

$$\sum_{n=0}^{p-1}\frac{D_n}{8^n} \equiv \sum_{k=0}^{p-1}\frac{p}{2k+1}\cdot\frac{f_k}{8^k} \pmod{p^3}. \tag{2.5}$$

It is known that ([JV])

$$f_k \equiv (-8)^k f_{p-1-k} \pmod p \quad \text{for} \quad k=0,1,\ldots,p-1. \tag{2.6}$$

Thus, from (2.3) and (2.5) we see that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{D_n}{8^n} &\equiv f_{\frac{p-1}{2}}8^{-\frac{p-1}{2}} + p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{1}{2k+1}\cdot\frac{f_k}{8^k}\\ &= 8^{-\frac{p-1}{2}}f_{\frac{p-1}{2}} + p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{1}{2(p-1-k)+1}\cdot\frac{f_{p-1-k}}{8^{p-1-k}}\\ &\equiv 8^{-\frac{p-1}{2}}f_{\frac{p-1}{2}} + p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{1}{-2k-1}\cdot\frac{f_k/(-8)^k}{8^{p-1-k}}\\ &\equiv 8^{-\frac{p-1}{2}}f_{\frac{p-1}{2}} - p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{1}{2k+1}(-1)^k f_k\\ &= ((-1)^{\frac{p-1}{2}} + 8^{-\frac{p-1}{2}})f_{\frac{p-1}{2}} - \sum_{k=0}^{p-1}\frac{p}{2k+1}(-1)^k f_k\\ &= \frac{1+(-8)^{\frac{p-1}{2}}}{8^{\frac{p-1}{2}}}f_{\frac{p-1}{2}} - \sum_{n=0}^{p-1}A_n \pmod{p^2}.\end{aligned}$$

If $p \equiv 5,7 \pmod 8$, then $(-8)^{(p-1)/2} \equiv -1 \pmod p$ and so $p \mid f_{\frac{p-1}{2}}$ by (2.6). Hence $(1+(-8)^{\frac{p-1}{2}})f_{\frac{p-1}{2}} \equiv 0 \pmod{p^2}$ and so

$$\sum_{n=0}^{p-1}\frac{D_n}{8^n} \equiv -\sum_{n=0}^{p-1}A_n \pmod{p^2} \quad \text{for} \quad p\equiv 5,7 \pmod 8. \tag{2.7}$$

Theorem 2.13. *Let p be a prime with $p \equiv 5 \pmod 6$. Then*

$$\sum_{n=0}^{p-1}\frac{D_n}{4^n} \equiv 0 \pmod{p^2}.$$

Proof. By [S8, Lemma 3.1],

$$D_n = \sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{3k}{k}\binom{n+k}{3k}4^{n-2k}.$$

Thus,

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{D_n}{4^n}&=\sum_{n=0}^{p-1}\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{3k}{k}\binom{n+k}{3k}4^{-2k}\\&=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{3k}{k}}{16^k}\sum_{n=2k}^{p-1}\binom{n+k}{3k}\\&=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{3k}{k}}{16^k}\sum_{r=0}^{p-1-2k}\binom{3k+r}{r}\\&=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{3k}{k}}{16^k}\sum_{r=0}^{p-1-2k}\binom{-3k-1}{r}(-1)^r\\&=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{3k}{k}}{16^k}\binom{p+k}{p-1-2k}=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{3k}{k}}{16^k}\binom{p+k}{3k+1}.\end{aligned}$$

For $1\le k\le\frac{p-1}{2}$ we see that

$$\begin{aligned}\binom{3k}{k}\binom{p+k}{3k+1}&=\frac{p}{3k+1}\cdot\frac{(p^2-1^2)\cdots(p^2-k^2)(p-(k+1))\cdots(p-2k)}{k!(2k)!}\\&\equiv\frac{p}{3k+1}\Big(1-p\sum_{r=k+1}^{2k}\frac{1}{r}\Big)\equiv\frac{p}{3k+1}\pmod{p^2}.\end{aligned}$$

Hence

$$\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\cdot\frac{p}{3k+1}\pmod{p^2}.$$

For $a\in\mathbb{Z}_p$ let

$$S_p(a)=\sum_{k=0}^{(p-1)/2}\binom{a}{k}\binom{-1-a}{k}\frac{1}{3k+1}.$$

By [S9, (3.2)],

$$(3a+1)S_p(a)-(3a-1)S_p(a-1)=2\binom{a-1}{\frac{p-1}{2}}\binom{-a-1}{\frac{p-1}{2}}.$$

For $a\not\equiv 0\pmod p$ we see that

$$\binom{a-1}{\frac{p-1}{2}}\binom{-a-1}{\frac{p-1}{2}}=\frac{(1^2-a^2)\cdots((\frac{p-1}{2})^2-a^2)}{\frac{p-1}{2}!^2}\equiv 0\pmod p.$$

Since $p\equiv 2\pmod 3$, we have $p\nmid 3k+1$ for $1\le k\le\frac{p-1}{2}$. Hence $S_p(a)\in\mathbb{Z}_p$ and so

$$S_p(a)\equiv\frac{3a-1}{3a+1}S_p(a-1)=\frac{2-6a}{-2-6a}S_p(a-1)\pmod p\quad\text{for}\quad a\not\equiv 0,-\frac{1}{3}\pmod p.$$

Therefore,

$$S_p\Big(-\frac{1}{2}\Big)\equiv\frac{5}{1}S_p\Big(-\frac{1}{2}-1\Big)\equiv\frac{5}{1}\cdot\frac{11}{7}S_p\Big(-\frac{1}{2}-2\Big)$$

$$\equiv \cdots \equiv \frac{5\cdot 11\cdots p}{1\cdot 7\cdots (p-4)} S_p\Big(-\frac{1}{2}-\frac{p+1}{6}\Big) \equiv 0 \pmod p.$$

Note that $\binom{-\frac{1}{2}}{k} = \binom{2k}{k}4^{-k}$. For $p \equiv 2 \pmod 3$ we see that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k(3k+1)} = \sum_{k=0}^{(p-1)/2} \binom{-1/2}{k}^2 \frac{1}{3k+1} = S_p\Big(-\frac{1}{2}\Big) \equiv 0 \pmod p$$

and so $\sum_{n=0}^{p-1} \frac{D_n}{4^n} \equiv 0 \pmod{p^2}$. This proves the theorem.

Theorem 2.14. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}\frac{a_k}{4^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1,3 \pmod 8 \text{ and so } p = x^2+2y^2,\\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Proof. Taking $u = 1$ in Theorem 2.6(ii) we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k}\frac{a_k}{4^k} \equiv \sum_{n=0}^{p-1} A_n \pmod p.$$

By [Su1, Corollary 1.2],

$$\sum_{n=0}^{p-1} A_n \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1,3 \pmod 8 \text{ and so } p = x^2+2y^2,\\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Thus the theorem is proved.

Remark 2.2 Let p be an odd prime, and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod p$. For conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{m^k} \pmod{p^2}$, see [Su3, Conjectures 7.8 and 7.9] and [S8, Conjectures 6.4-6.6].

Conjecture 2.1. *Let p be an odd prime. If $p \equiv 1,3 \pmod 8$ and so $p = x^2+2y^2$, then*

$$f_{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}}\big((3\cdot 2^{p-1}+1)x^2 - 2p\big) \pmod{p^2}$$

and

$$f_{\frac{p^2-1}{2}} \equiv 4x^4(3\cdot 2^{p-1}+1) - 16x^2p \pmod{p^2}.$$

Conjecture 2.2. *Let p be an odd prime. If $p \equiv 5,7 \pmod 8$, then*

$$f_{\frac{p^2-1}{2}} \equiv p^2 \pmod{p^3} \quad \text{and} \quad f_{\frac{p^r-1}{2}} \equiv 0 \pmod{p^r} \quad \text{for} \quad r \in \mathbb{Z}^+.$$

3. Congruences involving $\{S_n\}$

Recall that

$$S_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}x^k \quad (n = 0,1,2,\ldots)$$

and $S_n = S_n(1)$. From [G, (6.12)] we know that

$$S_n(-1) = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}(-1)^k = \begin{cases} 0 & \text{if } n \text{ is odd},\\ \binom{n}{n/2}^2 & \text{if } n \text{ is even}. \end{cases} \tag{3.1}$$

Using Maple and the Zeilberger algorithm we see that

$$n^2S_n = 4(3n^2-3n+1)S_{n-1} - 32(n-1)^2S_{n-2} \quad (n \ge 2).$$

Lemma 3.1. *For any nonnegative integer n we have*

$$S_n(-x) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k 4^{n-k} S_k(x).$$

Proof. Since $\binom{-\frac{1}{2}}{k} = \frac{\binom{2k}{k}}{(-4)^k}$, using Vandermonde's identity we see that for any nonnegative integer m,

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\binom{2k}{k}(-1)^k4^{m-k} &= 4^m\sum_{k=0}^{m}\binom{m}{m-k}\binom{-\frac{1}{2}}{k} = 4^m\binom{m-\frac{1}{2}}{m}\\ &= 4^m\cdot(-1)^m\binom{-\frac{1}{2}}{m} = \binom{2m}{m}.\end{aligned}$$

Note that $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$. From the above we see that

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k4^{n-k}S_k(x) &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k4^{n-k}\sum_{r=0}^{k}\binom{k}{r}\binom{2r}{r}\binom{2(k-r)}{k-r}x^r\\ &= \sum_{r=0}^{n}\binom{2r}{r}x^r\binom{n}{r}\sum_{k=r}^{n}\binom{n-r}{k-r}\binom{2(k-r)}{k-r}(-1)^k4^{n-k}\\ &= \sum_{r=0}^{n}\binom{n}{r}\binom{2r}{r}x^r(-1)^r\sum_{s=0}^{n-r}\binom{n-r}{s}\binom{2s}{s}(-1)^s4^{n-r-s}\\ &= \sum_{r=0}^{n}\binom{n}{r}\binom{2r}{r}(-x)^r\cdot\binom{2n-2r}{n-r} = S_n(-x).\end{aligned}$$

This proves the lemma.

Lemma 3.2. *For any nonnegative integer m we have*

$$\sum_{k=0}^{m}\binom{m}{k}S_k(x)n^{m-k} = \sum_{k=0}^{m}\binom{m}{k}(-1)^kS_k(-x)(n+4)^{m-k}$$

and so

$$\sum_{k=0}^{m}\binom{m}{k}S_kn^{m-k} = \sum_{k=0}^{[m/2]}\binom{m}{2k}\binom{2k}{k}^2(n+4)^{m-2k}.$$

Proof. Note that $\binom{m}{k}\binom{k}{r} = \binom{m}{r}\binom{m-r}{k-r}$. By Lemma 3.1,

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}S_k(x)n^{m-k} &= \sum_{k=0}^{m}\binom{m}{k}n^{m-k}\sum_{r=0}^{k}\binom{k}{r}(-1)^rS_r(-x)4^{k-r}\\ &= \sum_{r=0}^{m}(-1)^rS_r(-x)\sum_{k=r}^{m}\binom{m}{k}\binom{k}{r}4^{k-r}n^{m-k}\end{aligned}$$

$$= \sum_{r=0}^{m} \binom{m}{r} (-1)^r S_r(-x) n^{m-r} \sum_{k=r}^{m} \binom{m-r}{k-r} \Big(\frac{4}{n}\Big)^{k-r}$$

$$= \sum_{r=0}^{m} \binom{m}{r} (-1)^r S_r(-x) n^{m-r} \Big(1+\frac{4}{n}\Big)^{m-r}$$

$$= \sum_{r=0}^{m} \binom{m}{r} (-1)^r S_r(-x) (n+4)^{m-r}.$$

Taking $x = 1$ in the above formula and then applying (3.1) we deduce the remaining result.

If $\{c_n\}$ is a sequence satisfying

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k c_k = c_n \quad (n = 0, 1, 2, \ldots),$$

we say that $\{c_n\}$ is an even sequence. In [S1,S6] the author investigated the properties of even sequences.

Lemma 3.3. *Suppose that $\{c_n\}$ is an even sequence.*

(i) ([S7, Theorem 2.3]) *If n is odd, then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^k c_k = 0.$$

(ii) ([S7, Theorems 5.3 and 5.4]) *If p is a prime of the form $4k+3$ and $c_0, c_1, \ldots, c_{\frac{p-1}{2}} \in \mathbb{Z}_p$, then*

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \frac{c_k}{16^k} \equiv 0 \pmod{p^2} \quad \textit{and} \quad \sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{c_k}{2^k} \equiv 0 \pmod{p}.$$

Theorem 3.1. *Let n be a nonnegative integer. Then*

(i) $$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 4^{n-k} S_k = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \binom{n}{n/2}^2 & \text{if } n \text{ is even,} \end{cases}$$

(ii) $$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{S_k}{8^k} = \frac{S_n}{8^n},$$

(iii) $$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-8)^{n-k} S_k = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} \binom{n}{n/2}^3 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Taking $x = 1$ in Lemma 3.1 and then applying (3.1) we deduce part (i). By Lemma 3.2,

$$\sum_{k=0}^{n} \binom{n}{k} S_k m^{n-k}$$

$$= \sum_{k=0}^{[n/2]} \binom{n}{2k} \binom{2k}{k}^2 (m+4)^{n-2k} = (-1)^n \sum_{k=0}^{[n/2]} \binom{n}{2k} \binom{2k}{k}^2 (-m-4)^{n-2k}$$
$$= (-1)^n \sum_{k=0}^{n} \binom{n}{k} S_k (-m-8)^{n-k}.$$

That is,

$$\sum_{k=0}^{n} \binom{n}{k} S_k m^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k S_k (m+8)^{n-k}. \tag{3.2}$$

Putting $m = 0$ in (3.2) we obtain part (ii). By (ii), $\{\frac{S_n}{8^n}\}$ is an even sequence. Thus applying Lemma 3.3(i), for odd n we have

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-8)^{n-k} S_k = (-8)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^k \frac{S_k}{8^k} = 0.$$

Let

$$c_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-8)^{n-k} S_k.$$

Then $c_0 = 1$. Using Maple software doublesum.mpl and the method in [CHM] we find that $c_n = (\frac{4(n-1)}{n})^3 c_{n-2}$. When n is even we see that

$$\frac{(-1)^{n/2} \binom{n}{n/2}^3}{(-1)^{(n-2)/2} \binom{n-2}{(n-2)/2}^3} = \Big(\frac{4(n-1)}{n}\Big)^3.$$

Thus part (iii) holds for even n. The proof is now complete.

Lemma 3.4. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $x \not\equiv -1 \pmod p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{x}{8(1+x)^2}\Big)^k S_k \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \Big(-\frac{x^2}{64}\Big)^k \pmod p.$$

Proof. Taking $u = -x$ and $c_k = \frac{S_k}{(-8)^k}$ in Lemma 2.1 and then applying Theorem 3.1(iii) we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{-x}{(1+x)^2}\Big) \frac{S_k}{(-8)^k}$$
$$\equiv \sum_{n=0}^{p-1} (-x)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{S_k}{(-8)^k} = \sum_{k=0}^{(p-1)/2} (-x)^{2k} \cdot \frac{(-1)^k}{(-8)^{2k}} \binom{2k}{k}^3 \pmod p.$$

This yields the result.

Theorem 3.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{32^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1,3 \pmod 8 \text{ and so } p = x^2 + 2y^2, \\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Proof. Taking $x = 1$ in Lemma 3.4 we find that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{32^k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^3 \frac{1}{(-64)^k} \pmod p.$$

Now applying [S3, Theorems 3.3-3.4] we deduce the result.

Theorem 3.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{16^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1 \pmod 4 \text{ and so } p = x^2 + 4y^2, \\ 0 \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. From Theorem 3.1(ii) we know that $\{\frac{S_n}{8^n}\}$ is an even sequence. Thus applying Lemma 3.3(ii) we have $\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{16^k} \equiv 0 \pmod p$ for $p \equiv 3 \pmod 4$. Now assume $p \equiv 1 \pmod 4$ and so $p = x^2 + 4y^2$. Let $t \in \{1, 2, \ldots, \frac{p-1}{2}\}$ be given by $t^2 \equiv -1 \pmod p$. By Lemma 3.4,

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{16^k} &\equiv \sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{t}{8(1+t)^2}\Big)^k S_k \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^3 \Big(-\frac{t^2}{64}\Big)^k \\ &\equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^3 \frac{1}{64^k} \pmod p. \end{aligned}$$

It is well known that (see for example [Ah])

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^3 \frac{1}{64^k} \equiv 4x^2 - 2p \pmod {p^2}.$$

Thus $\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{16^k} \equiv 4x^2 \pmod p$, which completes the proof.

Theorem 3.4. *Let p be an odd prime and $q_p(2) = (2^{p-1} - 1)/p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{S_k}{128^k} \equiv \begin{cases} (-1)^{\frac{p-1}{4}} (8x^3 + 6x(2q_p(2)x^2 - 1)p) \pmod {p^2} \\ \qquad\qquad\qquad \text{if } p = x^2 + 4y^2 \equiv 1 \pmod 4 \text{ and } 4 \mid x - 1, \\ 0 \pmod {p^2} \qquad \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. It is clear that for $k \in \{0, 1, \ldots, \frac{p-1}{2}\}$,

$$\begin{aligned} &\binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k} \\ &= \binom{2k}{k}\binom{\frac{p-1}{2}+k}{2k} = \binom{2k}{k} \frac{(p^2-1^2)(p^2-3^2)\cdots(p^2-(2k-1)^2)}{2^{2k}\cdot(2k)!} \\ &\equiv \binom{2k}{k}(-1)^k \frac{1^2\cdot 3^2\cdots(2k-1)^2}{2^{2k}\cdot(2k)!} = \frac{\binom{2k}{k}^2}{(-16)^k} \pmod {p^2}. \end{aligned} \tag{3.3}$$

Thus, from Theorem 3.1(iii) we deduce that

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{S_k}{128^k} \equiv \sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k} \frac{S_k}{8^k}$$

$$= \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \\ (-1)^{\frac{p-1}{4}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}}^3 \pmod{p^2} & \text{if } p \equiv 1 \pmod 4. \end{cases}$$

By [CDE], for $p = x^2 + 4y^2 \equiv 1 \pmod 4$ with $4 \mid x - 1$,

$$\begin{aligned} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} &\equiv \frac{2^{p-1}+1}{2}\Big(2x - \frac{p}{2x}\Big) = \Big(1 + \frac{1}{2}q_p(2)p\Big)\Big(2x - \frac{p}{2x}\Big) \\ &\equiv 2x + p\Big(q_p(2)x - \frac{1}{2x}\Big) \pmod{p^2}. \end{aligned}$$

Thus,

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^3 \equiv \Big(2x + p\Big(q_p(2)x - \frac{1}{2x}\Big)\Big)^3 \equiv 8x^3 + 6x(2q_p(2)x^2 - 1)p \pmod{p^2}.$$

Now putting the above together we deduce the result.

For an odd prime p and $a \in \mathbb{Z}_p$ let $\langle a\rangle_p \in \{0, 1, \dots, p-1\}$ be given by $a \equiv \langle a\rangle_p \pmod p$.

Theorem 3.5. *Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$ and $\langle a\rangle_p \equiv 1 \pmod 2$. Then*

$$\sum_{k=0}^{p-1} \binom{a}{k}\binom{-1-a}{k}\frac{S_k}{8^k} \equiv 0 \pmod{p^2}.$$

In particular, for $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ we have

$$\begin{aligned} &\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{216^k} S_k \equiv 0 \pmod{p^2} \quad \text{for} \quad p \equiv 2 \pmod 3, \\ &\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{512^k} S_k \equiv 0 \pmod{p^2} \quad \text{for} \quad p \equiv 3 \pmod 4, \\ &\sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{3456^k} S_k \equiv 0 \pmod{p^2} \quad \text{for} \quad p \equiv 2 \pmod 3. \end{aligned}$$

Proof. This is immediate from Theorem 3.1(ii) and [S5, Theorem 2.4].

Theorem 3.6. *Let p be an odd prime, $n \in \mathbb{Z}_p$ and $n \not\equiv 0, -16 \pmod p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}\frac{S_k}{(n+16)^k} \equiv \Big(\frac{n(n+16)}{p}\Big)\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{n^{2k}} \pmod p.$$

Proof. Clearly $p \mid \binom{2k}{k}$ for $\frac{p}{2} < k < p$ and $p \mid \binom{2k}{k}\binom{4k}{2k}$ for $\frac{p}{4} < k < p$. Note that $\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod p$ for $0 \le k \le \frac{p-1}{2}$. By Lemma 3.2,

$$\sum_{k=0}^{p-1} \binom{2k}{k}\frac{S_k}{(n+16)^k}$$

$$\equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} S_k\Big(\frac{-4}{n+16}\Big)^k \equiv \Big(\frac{-n-16}{p}\Big)\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} S_k\Big(\frac{n+16}{-4}\Big)^{\frac{p-1}{2}-k}$$
$$= \Big(\frac{-n-16}{p}\Big)\sum_{k=0}^{[p/4]} \binom{\frac{p-1}{2}}{2k}\binom{2k}{k}^2\Big(-\frac{n}{4}\Big)^{\frac{p-1}{2}-2k}$$
$$\equiv \Big(\frac{-n(-n-16)}{p}\Big)\sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}}{(-4)^{2k}}\binom{2k}{k}^2\frac{1}{(-n/4)^{2k}} \equiv \Big(\frac{n(n+16)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{n^{2k}} \pmod p.$$

This proves the theorem.

Theorem 3.7. *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{7^k} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{25^k}$$
$$\equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,2,4 \pmod 7 \text{ and so } p=x^2+7y^2, \\ 0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7. \end{cases}$$

Proof. Taking $n=\pm 9$ in Theorem 3.6 and then applying [S3, Theorem 5.2] we deduce the result.

Theorem 3.8. *Let p be a prime such that $p\equiv 1,7,17,23 \pmod{24}$. Then*

$$\Big(\frac{3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{64^k} \equiv \Big(\frac{6}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(-32)^k}$$
$$\equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,7 \pmod{24} \text{ and so } p=x^2+6y^2, \\ 0 \pmod p & \text{if } p\equiv 17,23 \pmod{24}. \end{cases}$$

Proof. Taking $n=\pm 48$ in Theorem 3.6 and then applying [S3, Theorem 5.4] we deduce the result.

Theorem 3.9. *Let $p>5$ be a prime. Then*

$$\Big(\frac{2}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{800^k} \equiv \Big(\frac{3}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(-768)^k}$$
$$\equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,3 \pmod 8 \text{ and so } p=x^2+2y^2, \\ 0 \pmod p & \text{if } p\equiv 5,7 \pmod 8. \end{cases}$$

Proof. By [S8, Theorem 5.6],

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8, \\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8 \end{cases}$$

Now taking $n=\pm 28^2=\pm 784$ in Theorem 3.6 and then applying the above we obtain the result.

Theorem 3.10. *Let p be a prime such that $p\equiv 1,9 \pmod{10}$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{160^k} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(-128)^k}$$

$$\equiv \begin{cases} (\frac{2}{p})4x^2 \pmod p & \text{if } p \equiv 1,9,11,19 \pmod{40} \text{ and so } p = x^2+10y^2, \\ 0 \pmod p & \text{if } p \equiv 21,29,31,39 \pmod{40}. \end{cases}$$

Proof. Taking $n = \pm 144$ in Theorem 3.6 and then applying [S3, (5.9)] we deduce the result.

Theorem 3.11. *Let p be a prime such that $p \equiv \pm 1 \pmod 8$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{1600^k} \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{(-1568)^k}$$
$$\equiv \begin{cases} (\frac{-1}{p})4x^2 \pmod p & \text{if } p \equiv 1,3,4,5,9 \pmod{11} \text{ and so } p = x^2+22y^2, \\ 0 \pmod p & \text{if } p \equiv 2,6,7,8,10 \pmod{11}. \end{cases}$$

Proof. Taking $n = \pm 1584$ in Theorem 3.6 and then applying [S3, (5.9)] we deduce the result.

Theorem 3.12. *Let p be a prime such that $(\frac{p}{29}) = 1$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{156832^k} \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{(-156800)^k}$$
$$\equiv \begin{cases} (\frac{2}{p})4x^2 \pmod p & \text{if } p \equiv 1,3 \pmod 8 \text{ and so } p = x^2+58y^2, \\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Proof. Taking $n = \pm 396^2 = \pm 156816$ in Theorem 3.6 and then applying [S3, (5.9)] we deduce the result.

Theorem 3.13. *Let p be an odd prime, $n \in \mathbb{Z}_p$ and $n \not\equiv 0 \pmod p$.*

(i) *If $n \not\equiv 4 \pmod p$, then*

$$\sum_{k=0}^{p-1} \frac{S_k(x)}{n^k} \equiv \sum_{k=0}^{p-1} \frac{S_k(-x)}{(4-n)^k} \pmod p.$$

(ii) *If $n \not\equiv 16 \pmod p$, then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k(x)}{n^k} \equiv \Big(\frac{n(n-16)}{p}\Big) \sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k(-x)}{(16-n)^k} \pmod p.$$

Proof. Since $\binom{p-1}{k} \equiv (-1)^k \pmod p$ and $\binom{\frac{p-1}{2}}{k} \equiv \frac{\binom{2k}{k}}{(-4)^k} \pmod p$, taking $m = p-1$ and replacing n with $-n$ in Lemma 3.2 we deduce part (i), and taking $m = \frac{p-1}{2}$ and replacing n with $-\frac{n}{4}$ in Lemma 3.2 we deduce part (ii).

Conjecture 3.1. *Let p be an odd prime, $n \in \{\pm 156816, \pm 1584, \pm 784, \pm 144, \pm 48, \pm 9\}$ and $n \not\equiv 0, \pm 16 \pmod p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{(n+16)^k} \equiv \Big(\frac{n(n+16)}{p}\Big) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{n^{2k}} \pmod{p^2}.$$

Conjecture 3.2. *Let p be an odd prime. If $p \equiv 1, 3 \pmod 8$ and so $p = x^2 + 2y^2$, then*

$$S_{\frac{p-1}{2}} \equiv (5 \cdot 2^{p-1} - 1)x^2 - 2p \pmod{p^2}$$

and

$$S_{\frac{p^2-1}{2}} \equiv 4x^4(5 \cdot 2^{p-1} - 1) - 16x^2 p \pmod{p^2}.$$

Conjecture 3.3. *Let p be an odd prime. If $p \equiv 5, 7 \pmod 8$, then*

$$S_{\frac{p^2-1}{2}} \equiv p^2 \pmod{p^3} \quad \text{and} \quad S_{\frac{p^r-1}{2}} \equiv 0 \pmod{p^r} \quad \text{for} \quad r \in \mathbb{Z}^+.$$

Conjecture 3.4. *For $m = 2, 3, 4, \ldots$ we have*

$$S_m^2 < S_{m+1}S_{m-1} < \Big(1 + \frac{1}{m(m-1)}\Big)S_m^2.$$

4. Congruences involving $\{C_n\}$

For any nonnegative integer n define

$$C_n(x) = \sum_{k=0}^{[n/3]} \binom{2k}{k}\binom{3k}{k}\binom{n}{3k} x^{n-3k} \quad \text{and} \quad C_n = C_n(-3).$$

Lemma 4.1. *Let m be a nonnegative integer. Then*

$$\sum_{k=0}^{m} \binom{m}{k} C_k(x) n^{m-k} = C_m(x+n).$$

Proof. It is clear that

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k} C_k(x) n^{m-k} &= \sum_{k=0}^{m} \binom{m}{k} n^{m-k} \sum_{r=0}^{k} \binom{2r}{r}\binom{3r}{r}\binom{k}{3r} x^{k-3r} \\
&= \sum_{r=0}^{m} \binom{2r}{r}\binom{3r}{r} n^{m-3r} \sum_{k=r}^{m} \binom{m}{k}\binom{k}{3r}\Big(\frac{x}{n}\Big)^{k-3r} \\
&= \sum_{r=0}^{m} \binom{2r}{r}\binom{3r}{r} n^{m-3r} \sum_{k=3r}^{m} \binom{m}{3r}\binom{m-3r}{k-3r}\Big(\frac{x}{n}\Big)^{k-3r} \\
&= \sum_{r=0}^{m} \binom{2r}{r}\binom{3r}{r}\binom{m}{3r} n^{m-3r}\Big(1 + \frac{x}{n}\Big)^{m-3r} = C_m(x+n).
\end{aligned}$$

This proves the lemma.

Theorem 4.1. *Let p be an odd prime, $n, x \in \mathbb{Z}_p$ and $n(n+4x) \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{C_k(x)}{(n+4x)^k} \equiv \Big(\frac{n(n+4x)}{p}\Big) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{n^{3k}} \pmod p.$$

Proof. As $\binom{\frac{p-1}{2}}{k} \equiv \binom{2k}{k}4^{-k} \pmod p$ and $p \mid \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}$ for $\frac{p}{6} < k < p$, using Lemma 4.1 we see that

$$\begin{aligned}
&\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(x)}{(n+4x)^k}\\
&\equiv \sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}C_k(x)\Big(\frac{-4}{n+4x}\Big)^k \equiv \Big(\frac{-4(n+4x)}{p}\Big)\sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}C_k(x)\Big(\frac{n+4x}{-4}\Big)^{\frac{p-1}{2}-k}\\
&= \Big(\frac{-n-4x}{p}\Big)C_{\frac{p-1}{2}}\Big(-\frac{n}{4}\Big) = \Big(\frac{-n-4x}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{\frac{p-1}{2}}{3k}\Big(-\frac{n}{4}\Big)^{\frac{p-1}{2}-3k}\\
&\equiv \Big(\frac{n(n+4x)}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\frac{1}{(-4)^{3k}\cdot(-n/4)^{3k}}\\
&\equiv \Big(\frac{n(n+4x)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{n^{3k}} \pmod p.
\end{aligned}$$

This proves the theorem.

Theorem 4.2. *Let p be a prime, $p \neq 2, 3, 11$, $t \in \mathbb{Z}_p$ and $33 + 2t \not\equiv 0 \pmod p$. Then*

$$\Big(\frac{33+2t}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(t)}{(66+4t)^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p = x^2+4y^2 \equiv 1 \pmod 4,\\ 0 \pmod p & \text{if } p \equiv 3 \pmod 4.\end{cases}$$

Proof. Taking $n = 66$ and replacing x with t in Theorem 4.1 and then applying [S4, Theorem 4.3] we deduce the result.

Theorem 4.3. *Let $p > 5$ be a prime, $t \in \mathbb{Z}_p$ and $t \not\equiv -5 \pmod p$. Then*

$$\Big(\frac{-(5+t)}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(t)}{(20+4t)^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p = x^2+2y^2 \equiv 1,3 \pmod 8,\\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8.\end{cases}$$

Proof. Taking $n = 20$ and replacing x with t in Theorem 4.1 and then applying [S4, Theorem 4.4] we deduce the result.

Theorem 4.4. *Let $p > 7$ be a prime, $t \in \mathbb{Z}_p$ and $4t \not\equiv 15 \pmod p$. Then*

$$\Big(\frac{-15+4t}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(t)}{(-15+4t)^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p = x^2+7y^2 \equiv 1,2,4 \pmod 7,\\ 0 \pmod p & \text{if } p \equiv 3,5,6 \pmod 7.\end{cases}$$

Proof. Taking $n = -15$ and replacing x with t in Theorem 4.1 and then applying [S4, Theorem 4.7] we deduce the result.

Theorem 4.5. *Let $p > 7$ be a prime, $t \in \mathbb{Z}_p$ and $4t \not\equiv -255 \pmod p$. Then*

$$\Big(\frac{-255-4t}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(t)}{(255+4t)^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p = x^2+7y^2 \equiv 1,2,4 \pmod 7,\\ 0 \pmod p & \text{if } p \equiv 3,5,6 \pmod 7.\end{cases}$$

Proof. Taking $n = 255$ and replacing x with t in Theorem 4.1 and then applying [S4, Theorem 4.7] we deduce the result.

Theorem 4.6. *Let p be a prime, $p \neq 2,3,11$, $t \in \mathbb{Z}_p$ and $t \not\equiv 8 \pmod p$. Then*

$$\Big(\frac{t-8}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(t)}{(4t-32)^k} \equiv \begin{cases} x^2 \pmod p & \text{if } (\frac{p}{11}) = 1 \text{ and so } 4p = x^2+11y^2, \\ 0 \pmod p & \text{if } (\frac{p}{11}) = -1. \end{cases}$$

Proof. Taking $n = -32$ and replacing x with t in Theorem 4.1 and then applying [S4, Theorem 4.8] we deduce the result.

Theorem 4.7. *Let p be a prime, $p \neq 2,3,19$, $t \in \mathbb{Z}_p$ and $t \not\equiv 24 \pmod p$. Then*

$$\Big(\frac{t-24}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(t)}{(4t-96)^k} \equiv \begin{cases} x^2 \pmod p & \text{if } (\frac{p}{19}) = 1 \text{ and so } 4p = x^2+19y^2, \\ 0 \pmod p & \text{if } (\frac{p}{19}) = -1. \end{cases}$$

Proof. Taking $n = -96$ and replacing x with t in Theorem 4.1 and then applying [S4, Theorem 4.9] we deduce the result.

Using Theorem 4.1 and [S4, Theorem 4.9] one can also deduce the following results.

Theorem 4.8. *Let p be a prime, $p \neq 2,3,5,43$, $t \in \mathbb{Z}_p$ and $t \not\equiv 240 \pmod p$. Then*

$$\Big(\frac{t-240}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(t)}{(4t-960)^k} \equiv \begin{cases} x^2 \pmod p & \text{if } (\frac{p}{43}) = 1 \text{ and so } 4p = x^2+43y^2, \\ 0 \pmod p & \text{if } (\frac{p}{43}) = -1. \end{cases}$$

Theorem 4.9. *Let p be a prime, $p \neq 2,3,5,11,67$, $t \in \mathbb{Z}_p$ and $t \not\equiv 1320 \pmod p$. Then*

$$\Big(\frac{t-1320}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(t)}{(4t-5280)^k} \equiv \begin{cases} x^2 \pmod p & \text{if } (\frac{p}{67}) = 1 \text{ and so } 4p = x^2+67y^2, \\ 0 \pmod p & \text{if } (\frac{p}{67}) = -1. \end{cases}$$

Theorem 4.10. *Let p be a prime, $p \neq 2,3,5,23,29,163$, $t \in \mathbb{Z}_p$ and $t \not\equiv 160080 \pmod p$. Then*

$$\Big(\frac{t-160080}{p}\Big)\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k(t)}{(4t-640320)^k}$$
$$\equiv \begin{cases} x^2 \pmod p & \text{if } (\frac{p}{163}) = 1 \text{ and so } 4p = x^2+163y^2, \\ 0 \pmod p & \text{if } (\frac{p}{163}) = -1. \end{cases}$$

Conjecture 4.1. *Let p be an odd prime, $n \in \{-640320, -5280, -960, -96, -32, -15, 20, 66, 255\}$ and $n(n-12) \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{C_k}{(n-12)^k} \equiv \Big(\frac{n(n-12)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{n^{3k}} \pmod{p^2}.$$

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